

## lecture

prove  $(A \times B)^{-1} = B^{-1} \times A^{-1}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix}$$

proof  $A \times B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 7+8 & 6+6 \\ 21+16 & 19+12 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 37 & 30 \end{bmatrix}$$

$$(A \times B)^{-1} = \frac{1}{(15)(30) - (12)(37)} \begin{bmatrix} 30 & -12 \\ -37 & 15 \end{bmatrix}$$
$$= \frac{1}{6} \begin{bmatrix} 30 & -12 \\ -37 & 15 \end{bmatrix}$$

•  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{21-24} \begin{bmatrix} 3 & -6 \\ -4 & 7 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 3 & -6 \\ -4 & 7 \end{bmatrix}$$

$$B^{-1} A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 30 & -12 \\ -37 & 15 \end{bmatrix}$$



[2] Inverse  $\Rightarrow$  Row elimination

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Jalga

$$[A/I] \Rightarrow [I/A^{-1}]$$

$$r_1 - r_2 \rightarrow r_2 \quad \left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right]$$

$$2r_2 + r_1 \rightarrow r_1 \quad \left[ \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{array} \right]$$

$$-r_2 \rightarrow r_2 \quad \left[ \begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & -1 & -1 & 1 \end{array} \right] = \left[ \begin{array}{cc|cc} 1 & -1 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

$= I A^{-1}$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Ex Find  $A^{-1}$ :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \quad \boxed{3 \times 3}$$

$$|A| = \begin{bmatrix} \oplus & \ominus & \oplus \\ 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} = 1(5 \times 8 - 0 \times 3) - 2[2 \times 8 - 1 \times 3] + 3[2 \times 0 - 1 \times 5]$$

$$= 40 - 2(16 - 3) + 3(-5) = 40 - 26 - 15 = 1$$



$$A^{-1} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \times \frac{1}{-1}$$

$$= - \begin{bmatrix} +(5 \times 8 - 0 \times 3) & -2(2 \times 8 - 0 \times 3) & +(2 \times 3 - 5 \times 3) \\ -(2 \times 8 - 1 \times 3) & +(1 \times 8 - 1 \times 3) & -(1 \times 3 - 2 \times 3) \\ +(2 \times 0 - 1 \times 5) & -(1 \times 2 - 2 \times 3) & +(1 \times 5 - 2 \times 2) \end{bmatrix}$$

$$= - \begin{bmatrix} 40 & -16 & 9 \\ -13 & 5 & -3 \\ -5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -40 & 16 & -9 \\ 13 & -5 & 3 \\ 5 & -2 & -1 \end{bmatrix}$$

Find  $A^{-1}$ :

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2(1-0) - 0(-1+0) - 1(0+1) \\ = 2 - 1 \\ = 1$$

$$A^{-1} = \begin{bmatrix} +(1-0) & -(0-0) & +(0+1) \\ -(-1-0) & +(2-1) & -(0-1) \\ +(0+1) & -(0-0) & +(2+0) \end{bmatrix}$$

$$[A/I] \Rightarrow [I/A^{-1}] \quad \text{Another}$$



$$\begin{array}{l}
 r_1 + 2r_2 \rightarrow r_2 \\
 r_1 + 2r_3 \rightarrow r_3
 \end{array}
 \left[ \begin{array}{ccc|ccc}
 2 & 0 & -1 & 1 & 0 & 0 \\
 -1 & 1 & 0 & 0 & 1 & 0 \\
 -1 & 0 & 1 & 0 & 0 & 1
 \end{array} \right]$$

$$\begin{array}{l}
 \cancel{r_1} \quad r_3 + r_1 \rightarrow r_1 \\
 \cancel{r_2} \quad r_3 + r_2 \rightarrow r_2
 \end{array}
 \left[ \begin{array}{ccc|ccc}
 2 & 0 & -1 & 1 & 0 & 0 \\
 0 & 2 & -1 & 1 & 2 & 0 \\
 0 & 0 & 1 & 1 & 0 & 2
 \end{array} \right]$$

$$\begin{array}{l}
 \frac{r_1}{2} \rightarrow r_1 \\
 \frac{r_2}{2} \rightarrow r_2
 \end{array}
 \left[ \begin{array}{ccc|ccc}
 2 & 0 & 0 & 2 & 0 & 2 \\
 0 & 2 & 0 & 2 & 2 & 2 \\
 0 & 0 & 1 & 1 & 0 & 2
 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc}
 1 & 0 & 0 & 1 & 0 & 1 \\
 0 & 1 & 0 & 1 & 1 & 1 \\
 0 & 0 & 1 & 1 & 0 & 2
 \end{array} \right] \quad A^{-1}$$

Using Row Operations to find  $A^{-1}$

$$A = \left[ \begin{array}{ccc|ccc}
 1 & 2 & 3 & 1 & 0 & 0 \\
 2 & 5 & 3 & 0 & 1 & 0 \\
 1 & 0 & 8 & 0 & 0 & 1
 \end{array} \right]$$

$$\begin{array}{l}
 -2r_1 + r_2 \rightarrow r_2 \\
 r_3 - r_1 \rightarrow r_3
 \end{array}
 \left[ \begin{array}{ccc|ccc}
 1 & 2 & 3 & 1 & 0 & 0 \\
 2 & 5 & 3 & 0 & 1 & 0 \\
 1 & 0 & 8 & 0 & 0 & 1
 \end{array} \right]$$



$$\begin{array}{l} -r_2 + r_1 \rightarrow r_1 \\ 2r_2 + r_3 \rightarrow r_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} 9r_3 + r_1 \rightarrow r_1 \\ -3r_3 + r_2 \rightarrow r_2 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & 2 & 1 \end{array} \right]$$

$$-r_3 \rightarrow r_3 \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & -13 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ -13 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$$

Ex Show that the matrix A is Not Invertible by 2 different Method

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{vmatrix} = 1(20+2) - 6(10-1) + 4(4+4) \\ &= 22 - 54 + 32 = 54 - 54 = 0 \end{aligned}$$



$$\therefore A^{-1} = \frac{1}{0} \Rightarrow \infty$$

$$\begin{array}{l} -2r_1 + r_2 \rightarrow r_2 \\ r_1 + r_3 \rightarrow r_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array} \right]$$

Not invertable  $r_2 + r_3 \rightarrow r_3$

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array} \right]$$

Not invertable!

Solve the system of linear equation by using the inverse of the matrix

Ex Solve the system of equation by hand the inverse

$$x + y = 3$$

$$2x + y = 5$$

Solution

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$



$$\begin{matrix} A & x & = & b \\ 2 \times 2 & 2 \times 1 & & 2 \times 1 \end{matrix}$$

$$A^{-1} A x = A^{-1} b, \quad I x = A^{-1} b$$

$$x = A^{-1} b \Rightarrow A^{-1} = \frac{-1}{1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} 3 & +5 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\therefore x = 2 \quad y = 1$$

Solve the system of equation by using the inverse

$$x_1 + 2x_2 + 3x_3 = 5$$

$$2x_1 + 5x_2 + 3x_3 = 3$$

$$x_1 + \quad + 8x_3 = 17$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$\begin{matrix} A & & x & & b \end{matrix}$

$$Ax = b \quad A^{-1} Ax = A^{-1} b \quad \therefore \boxed{x = A^{-1} b}$$



$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ \cancel{18} 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -40 & 16 & 9 \\ \cancel{18} 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} -200 + 201 \\ 65 - 66 \\ 25 - 23 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Ex 8 p 46

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Powers of a Matrix

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}$$



$$\bullet A^3 = A \times A \times A = A^2 \times A =$$

$$= \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = A^{-1} A^{-1} A^{-1} = (A^3)^{-1}$$

$$A^{-3} = \frac{1}{11 \times 41 - 30 \times 15} \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Laws of exponents :

if  $A$  is Square Matrix,  $r$  &  $s$  are integers

$$A^r A^s = A^{r+s} \quad (A^r)^s = A^{rs}$$

Polynomial Expressions involving Matrices

$$\text{if } p(x) = 2x^2 - 3x + 4 \quad \& \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

find  $p(A) = ??$  then find  $A^{-1}$

$$p(x) = 2x^2 - 3x + 4$$

$$p(A) = 2A^2 - 3A + 4I$$

$$p(A) = 2 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$= 2 \begin{bmatrix} 1 & 5 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$p(A) = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix} = 2A^2 - 3A + 4I$$

$$A^{-1} \left[ \begin{array}{c} \downarrow \downarrow \downarrow \\ B = 2A^2 - 3A + 4I \end{array} \right]$$

$$A^{-1}B = 2A^{-1}A^2 - 3A^{-1}A + 4A^{-1}I$$

$$A^{-1}B = 2 \underbrace{A^{-1}A}_I A - 3I + 4A^{-1}$$

$$A^{-1}B = 2A - 3I + 4A^{-1}$$

$$p(A) = \begin{bmatrix} 19 & 2 \\ 0 & 13 \end{bmatrix}$$

$$A^{-1}B - 4A^{-1} = 2A - 3I$$

$$A^{-1} [B - 4I] = 2A - 3I$$



$$A^{-1} \left\{ \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right\} =$$

$$2 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}^{-1} = \frac{1}{45} \begin{bmatrix} -5 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ 0 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{45} \begin{bmatrix} -45 & 30 \\ 0 & 15 \end{bmatrix}$$



### Ex 3 p 64

What Conditions must  $b_1$  &  $b_2$  &  $b_3$  Satisfy in order  
For the system of equations:

$$x_1 + 2x_2 + 2x_3 = b_1$$

$$x_1 + \quad + x_3 = b_2$$

$$2x_1 + x_2 + 3x_3 = b_3$$

To Be Consistent ??

One Solution

more than 1 Solution

$$\begin{array}{l} r_1 - r_2 \rightarrow r_2 \\ -2r_1 + r_3 \rightarrow r_3 \end{array} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array} \right]$$

$$r_2 + r_3 \rightarrow r_3 \left[ \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_1 - b_2 + b_3 - 2b_1 \end{array} \right]$$

$$x_2 + x_3 = b_1 - b_2$$

$$x_3 = t$$



$$x_2 = -t + b_1 - b_2$$

$$x_1 = -t + b_2$$

Assignment

dis dp

$$x + 2y - 3z = 4$$

$$3x - y + 5z = 2$$

$$4x + y + (a^2 - 14)z = a + 2$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2 - 14 & a + 2 \end{array} \right]$$

$$\Rightarrow \begin{array}{l} -3r_1 + r_2 \rightarrow r_2 \\ -4r_1 + r_3 \rightarrow r_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2 - 2 & a - 14 \end{array} \right]$$

No Sol

One Sol

More than 1 Sol

$$a^2 - 16 = 0$$

$$R = \{4, -4\}$$

$$a^2 - 16 = 0$$

$$a = 4$$

$$a - 4 \neq 0$$

$$a = 4$$

$$a \neq 4$$

$$-7y + 14z = -10 \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{8}{7} \\ \frac{10}{7} \\ 0 \end{bmatrix}$$



$$7y = 10 - 14t$$

$$y = \frac{10}{7} - 2t$$

$$x + 2y - 3z = 4$$

$$x = \frac{-3}{7} + 4t + 3t + 4$$

$$x = \frac{8}{7} + 7t$$

$$12) a \quad 2x_1 - 3x_2 + 4x_3 - x_4 = 0$$

$$7x_1 + x_2 - 8x_3 + 9x_4 = 0$$

$$2x_1 + 8x_2 + x_3 - x_4 = 0$$

$$\left[ \begin{array}{cccc|c} 2 & -3 & 4 & -1 & 0 \\ 7 & 1 & -8 & 9 & 0 \\ 2 & 8 & 1 & -1 & 0 \end{array} \right] \Rightarrow \begin{array}{l} 7r_1 - 2r_2 \rightarrow r_3 \\ r_1 - r_3 \rightarrow r_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 2 & -3 & 4 & -1 & 0 \\ 0 & -23 & 44 & -25 & 0 \\ 0 & -11 & 3 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 2 & -3 & 4 & -1 & 0 \\ 0 & -22 & 44 & -25 & 0 \\ 0 & 0 & 45 & -275 & 0 \end{array} \right]$$